# Stability of the Einstein static universe in modified Gauss-Bonnet gravity

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We analyze the stability of the Einstein static universe by considering linear homogeneous perturbations in the context of $f(\mathcal{G})$ modified Gauss-Bonnet theories of gravity. By considering a generic form of $f(\mathcal{G})$, the stability region of the Einstein static universe is parameterized by the linear equation of state parameter $w = p/\rho$ and the second derivative $f''(\mathcal{G})$ of the Gauss-Bonnet term.

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## I. INTRODUCTION

Cosmology is said to be thriving in a golden age, where a central theme is the perplexing fact that the Universe is undergoing an accelerating expansion [1]. The latter, one of the most important and challenging current problems in cosmology, represents a new imbalance in the governing gravitational equations. Historically, physics has addressed such imbalances by either identifying sources that were previously unaccounted for, or by altering the governing equations. The cause of this acceleration still remains an open and tantalizing question. Although the introduction of a cosmological constant into the field equations seems to be the simplest theoretical approach to generate a phase of accelerated expansion, several alternative candidates have been proposed in the literature, ranging from dynamical dark energy models to modified theories of gravity. Amongst the latter, models generalizing the Einstein-Hilbert action have been proposed.

The Einstein field equations of General Relativity were first derived from an action principle by Hilbert, by adopting a linear functional of the scalar curvature, $R$, in the gravitational Lagrangian density. However, there are no *a priori* reasons to restrict the gravitational Lagrangian to this form, and indeed several generalizations of the Einstein-Hilbert Lagrangian have been proposed, including "quadratic Lagrangians", involving second order curvature invariants such as $R^2$, $R_{\mu\nu}R^{\mu\nu}$, $R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$, $\varepsilon^{\alpha\beta\mu\nu}R_{\alpha\beta\gamma\delta}R^{\gamma\delta}_{\mu\nu}$, $C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu}$, etc [2] (see Ref. [3] for a recent review). The physical motivations for these modifications of gravity were related to the possibility of a more realistic representation of the gravitational fields near curvature singularities and to create some first order approximation for the quantum theory of gravitational fields.

In considering alternative higher-order gravity theories, one is liable to be motivated in pursuing models consistent and inspired by several candidates of a fundamental theory of quantum gravity. In this context, it may be possible that unusual gravity-matter couplings predicted by string/M-theory may become important at the recent low-curvature Universe. For instance, one may couple a scalar field not only with the curvature scalar, as in scalar-tensor theories, but also with higher order curvature invariants. Indeed, motivations from string/M-theory predict that scalar field couplings with the Gauss-Bonnet invariant $\mathcal{G}$ are important in the appearance of non-singular early time cosmologies [4]. It is also possible to apply these motivations to the late-time Universe in an effective Gauss-Bonnet dark energy model [5].

In the context of $f(\mathcal{G})$ modified theories of gravity, we explore the stability of the Einstein static universe in this work. This is motivated by the possibility that the universe might have started out in an asymptotically Einstein static state, in the inflationary universe context [6]. On the other hand, the Einstein cosmos has always been of great interest in various gravitational theories. In general relativity for instance, generalizations with non-constant pressure have been analyzed in [7]. In brane world models, the Einstein static universe was investigated in [8], while its generalization within Einstein-Cartan theory can be found in [9], and in loop quantum cosmology, we refer the reader to [10].

In the context of $f(R)$ modified theories of gravity, the stability of the Einstein static universe was also analyzed by considering homogeneous perturbations [11]. By considering specific forms of $f(R)$, the stability regions of the solutions were parameterized by a linear equation of state parameter $w = p/\rho$. Contrary to classical general relativity, it was found that in $f(R)$ gravity a stable Einstein cosmos with a positive cosmological constant does indeed exist. Thus, in principle, modifications in $f(R)$ gravity stabilize solutions which are unstable in general relativity. Furthermore, in [12] it was found that only one class of $f(R)$ theories admits an Einstein static model, and that this class is neutrally stable with respect to vector and tensor perturbations for all equations of state on all

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scales. These results are apparently contradictory with those of Ref. [11]. However, in a recent work, homogeneous and inhomogeneous scalar perturbations in the Einstein static solutions were analyzed [13], consequently reconciling both of the above works.

This paper is outlined in the following manner: In Section II, we briefly review modified Gauss-Bonnet gravity, for self-completeness and self-consistency, and present the respective field equations. In Section III, we consider a generic form of $f(\mathcal{G})$, and analyze the stability of the solutions, by considering homogeneous perturbations around the Einstein static universe, and deduce the respective stability regions. The latter are given in terms of the linear equation of state parameter $w = p/\rho$ and the unperturbed energy density $\rho_0$. Finally, in Section IV we present our conclusions. Throughout this work, we consider the following units $c = G = 1$.

## II. MODIFIED GAUSS-BONNET GRAVITY AND FIELD EQUATIONS

An interesting alternative gravitational theory is modified Gauss-Bonnet gravity, which is given by the following action

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa} + f(\mathcal{G})\right] + S_M(g_{\mu\nu}, \psi)\,, \qquad (1)$$

where $\kappa = 8\pi$. $f(\mathcal{G})$ is an arbitrary function of the Gauss-Bonnet invariant, which is in turn given by

$$\mathcal{G} \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\,. \qquad (2)$$

$S_M(g_{\mu\nu}, \psi)$ is the matter action, defined as $S_M = \int d^4x\sqrt{-g}\;\; \mathcal{L}_m(g_{\mu\nu}, \psi)$, where $\mathcal{L}_m$ is the matter Lagrangian density, in which matter is minimally coupled to the metric $g_{\mu\nu}$ and $\psi$ collectively denotes the matter fields. The matter stress-energy tensor, $T^{(m)}_{\mu\nu}$, is defined as $T^{(m)}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\,\mathcal{L}_m)}{\delta(g^{\mu\nu})}$. Thus, using the diffeomorphism invariance of $S_M(g_{\mu\nu}, \psi)$ yields the covariant conservation of the stress-energy tensor, $\nabla^\mu T^{(m)}_{\mu\nu} = 0$.

Let us for the moment assume the function $f(\mathcal{G})$ to be expanded around the origin so that we may write

$$f(\mathcal{G}) = f(0) + f'(0)\mathcal{G} + \frac{f''(0)}{2!}\mathcal{G}^2 + O(\mathcal{G}^3)\,. \qquad (3)$$

If we insert this representation back into the Lagrangian, we note that $f(0)$ acts like an effective cosmological constant. The term $f'(0)$ would not be present in the field equations because this term is linear in the Gauss-Bonnet invariant and linear topological terms do not contribute to the equations of motion. The first non-trivial term to be expected is $f''(0)$. Moreover, as we are interested in linear perturbation theory, we expect only the term $f''(0)$ to characterize deviations from general relativity. Hence, we anticipate that the general relativistic limit takes the form $f''(0) \to 0$.

Modified Gauss-Bonnet gravity has been extensively analyzed in the literature, and rather than review all of its intricate details here, we refer the reader to Refs. [14, 15, 16]. Varying the action Eq. (1) with respect to $g^{\mu\nu}$, one obtains the gravitational field equations in the following form $G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = \kappa\, T^{\mu\nu}_{\rm eff}$, where the effective stress-energy tensor is defined as $T^{\mu\nu}_{\rm eff} = T^{\mu\nu}_{(m)} + T^{\mu\nu}_{\rm GB}$. The effective Gauss-Bonnet curvature stress-energy term is given by

$$\begin{aligned} T^{\mu\nu}_{\rm GB} = {}& \frac{1}{2}g^{\mu\nu}f(\mathcal{G}) - 2f'(\mathcal{G})RR^{\mu\nu} + 4f'(\mathcal{G})R^{\mu}{}_{\rho}R^{\nu\rho} - 2f'(\mathcal{G})R^{\mu\rho\sigma\tau}R^{\nu}{}_{\rho\sigma\tau} - 4f'(\mathcal{G})R^{\mu\rho\sigma\nu}R_{\rho\sigma} + 2[\nabla^\mu\nabla^\nu f'(\mathcal{G})]R \\ & - 2g^{\mu\nu}[\nabla^2 f'(\mathcal{G})]R - 4[\nabla_\rho\nabla^\mu f'(\mathcal{G})]R^{\nu\rho} - 4[\nabla_\rho\nabla^\nu f'(\mathcal{G})]R^{\mu\rho} + 4[\nabla^2 f'(\mathcal{G})]R^{\mu\nu} \\ & + 4g^{\mu\nu}[\nabla_\rho\nabla_\sigma f'(\mathcal{G})]R^{\rho\sigma} - 4[\nabla_\rho\nabla_\sigma f'(\mathcal{G})]R^{\mu\rho\nu\sigma}\,, \end{aligned} \qquad (4)$$

where the definition $f'(\mathcal{G}) \equiv df(\mathcal{G})/d\mathcal{G}$ is considered for notational simplicity.

## III. THE EINSTEIN STATIC UNIVERSE IN $f(\mathcal{G})$ GRAVITY AND PERTURBATIONS

### A. Metric and field equations

Consider the metric given by

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-r^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]. \qquad (5)$$

The Gauss-Bonnet invariant in provided by

$$\mathcal{G} = 24\frac{1+a'^2}{a^2}\frac{a''}{a}\,, \tag{6}$$

where the prime, in this context, denotes differentiation with respect to cosmological time $t$.

For the Einstein static universe, $a = a_0 = \text{const}$, the Ricci scalar becomes $R = 6/a_0^2$, and the Gauss-Bonnet invariant is trivially given by $\mathcal{G}(a_0) = 0$. The field equations in this case take the following form

$$\frac{3}{a_0^2} = \kappa\left(\rho_0 - \frac{f(0)}{2}\right)\,, \qquad -\frac{1}{a_0^2} = \kappa\left(p_0 + \frac{f(0)}{2}\right)\,, \tag{7}$$

where $\rho_0$ and $p_0$ are the unperturbed energy density and isotropic pressure, respectively. It should be noted that the modified Gauss-Bonnet term $f(0)$ acts like an effective cosmological constant to the background field equations, as noted above.

### B. Linear homogeneous perturbations

In what follows, we analyze the stability against linear homogeneous perturbations around the Einstein static universe given in Eqs. (7). Thus, we introduce perturbations in the energy density and the metric scale factor which only depend on time

$$\rho(t) = \rho_0\left(1+\delta\rho(t)\right)\,, \qquad a(t) = a_0\left(1+\delta a(t)\right)\,. \tag{8}$$

Subsequently, we consider a linear equation of state, $p(t) = w\rho(t)$, linearize the perturbed field equations and analyze the dynamics of the solutions.

Despite the Gauss-Bonnet invariant being trivial at the background level, the perturbations have an effect. From Eq. (6) it follows that

$$\delta\mathcal{G} = 24\frac{k}{a_0^2}\frac{\delta a''}{a_0}\,, \tag{9}$$

in linear order. Therefore, using $f(\mathcal{G}+\delta\mathcal{G}) = f(\mathcal{G}) + f'(\mathcal{G})\delta\mathcal{G}$ plus higher order terms, we find for linear perturbations

$$f(\mathcal{G}) = f(0) + 24f'(0)\frac{k}{a_0^2}\frac{\delta a''}{a_0}\,, \tag{10}$$

where we assumed that $f$ is an analytic function.

We now insert the perturbations (8) into the full field equations and linearize. This provides the following two equations

$$\frac{6}{a_0^2}\delta a + \kappa\rho_o\delta\rho = 0 \tag{11}$$

$$\frac{2}{a_0^2}\delta a - 2\delta a'' - \kappa\rho_0 w\delta\rho - 96\kappa\frac{f''(0)}{a_0^4}\delta a^{(4)} = 0 \tag{12}$$

It should be noted that these perturbation equations do not contain contributions of the form $f'(0)$, as expected. The first equation relates the perturbations in the scale factor to the density perturbations. Next, we can eliminate $\delta\rho$ from the second equation using the first and we arrive at the following fourth order perturbation equations for the perturbed scale factor

$$\frac{2}{a_0^2}(1+3w)\delta a - 2\delta a'' - 96\kappa\frac{f''(0)}{a_0^4}\delta a^{(4)} = 0\,. \tag{13}$$

By virtue of the background equations we have

$$\frac{2}{a_0^2} = \kappa(1+w)\rho_0\,, \tag{14}$$

and therefore we finally arrive at the following perturbation equation

$$24\kappa^3\rho_0^2(1+w)^2 f''(0)\delta a^{(4)}(t) + 2\ddot{\delta a}(t) - \kappa\rho_0(1+w)(1+3w)\delta a(t) = 0\,. \tag{15}$$

### C. Stability regions

Using the standard ansatz $\delta a(t) = C\exp(\omega t)$, where $C$ and $\omega$ are constants, we find that Eq. (15) provides as solutions the following four frequencies of the small perturbations

$$\omega_\pm^2 = \frac{-1 \pm \sqrt{1+24\kappa^4\rho_0^3(1+w)^3(1+3w)f''(0)}}{24\kappa^3\rho_0^2(1+w)^2 f''(0)}\,. \tag{16}$$

As expected, in the limit $f''(0) \to 0$ the frequencies $\omega_+^2$ lead to the general relativistic result

$$\omega_{\rm GR}^2 = \frac{\kappa\rho_0}{2}(1+w)(1+3w)\,, \tag{17}$$

while the frequencies $\omega_-^2$ become formally infinite in this limit, which is simply an artifact of the reduction of fourth-order to second order gravity.

The perturbations $\delta a$ are stable if the frequencies $\omega$ are purely complex, which implies that the stability is equivalent to the two conditions $\omega_\pm^2 < 0$. In order to simplify the notation, let us introduce the new parameter $\alpha = 24\kappa^4\rho_0^3 f''(0)$, so that the perturbations are stable for the following three parameter regions.

First, there is a region analogous to the classical general relativistic stability region given by

$$-\frac{1}{3} > w > -1\,, \qquad -\frac{1}{(1+w)^3(1+3w)} \geq \alpha > 0\,. \tag{18}$$

The second region covers the range where the equation of state is larger due to the presence of the Gauss-Bonnet term, and is provided by

$$w > -\frac{1}{3}\,, \qquad 0 > \alpha \geq -\frac{1}{(1+w)^3(1+3w)}\,. \tag{19}$$

Finally, there exits a stable region where the equation of state is less than minus one. We denote this the phantom region, and the stability regions are given by

$$w < -1\,, \qquad 0 > \alpha \geq -\frac{1}{(1+w)^3(1+3w)}\,. \tag{20}$$

These three regions are depicted in Fig. 1 and it is evident that there exits stable modes for all equation of state parameters $w$ if $\alpha$ is chosen appropriately. We emphasize that this includes positive values of the equation of state. It is interesting to note that the negative values of $\alpha = 24\kappa^4\rho_0^3 f''(0)$, imply that $f''(0) < 0$.

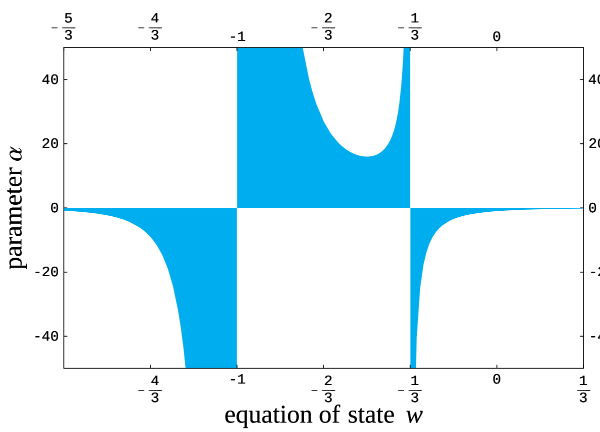


FIG. 1: Regions of stability in the $(w, \alpha)$ parameter space for homogeneous perturbations of Einstein static universes. The left is the phantom region, the middle part is the general relativistic analogue region and the right part corresponds to a normal matter region. Note that there exist stable modes for all $w$, provided $\alpha$ is chosen appropriately.

## IV. SUMMARY AND DISCUSSION

The Einstein static universe has recently been revived as the asymptotic origin of an emergent universe, namely, as an inflationary cosmology without a singularity [6]. The role of positive curvature, negligible at late times, is crucial in the early universe, as it allows these cosmologies to inflate and later reheat to a hot big-bang epoch. An attractive feature of these cosmological models is the absence of a singularity, of an 'initial time', of the horizon problem, and the quantum regime can even be avoided. Furthermore, the Einstein static universe was found to be neutrally stable against inhomogeneous linear vector and tensor perturbations, and against scalar density perturbations provided that the speed of sound satisfies $c_s^2 > 1/5$ [17]. Further issues related to the stability of the Einstein static universe may be found in Ref. [18].

In this work we have analyzed linear homogeneous perturbations around the Einstein static universe in the context of $f(\mathcal{G})$ modified theories of gravity. In particular, perturbations in the energy density and the metric scale factor were introduced, a linear equation of state, $p(t) = w\rho(t)$, was considered, and finally the linearized perturbed field equations and the dynamics of the solutions were analyzed. It was shown that stable modes for all equation of state parameters $w$ exist, if the parameter $\alpha$ is chosen appropriately. Thus, as in Refs. [11, 12, 13] our results show that perturbation theory of modified theories of gravity present a richer stability/instability structure that in general relativity. Finally, it is of interest to extend our results to inhomogeneous perturbations in the spirit of Ref. [13], and to include the canonical scalar field case. Work along these lines is presently underway.

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